

Nonstationary increments, scaling distributions, and variable diffusion processes in financial markets

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Fat-tailed distributions have been reported in fluctuations of financial markets for more than a decade. Sliding interval techniques used in these studies implicitly assume that the underlying stochastic process has stationary increments. Through an analysis of intraday increments, we explicitly show that this assumption is invalid for the Euro–Dollar exchange rate. We find several time intervals during the day where the standard deviation of increments exhibits power law behavior in time. Stochastic dynamics during these intervals is shown to be given by diffusion processes with a diffusion coefficient that depends on time and the exchange rate. We introduce methods to evaluate the dynamical scaling index and the scaling function empirically. In general, the scaling index is significantly smaller than previously reported values close to 0.5. We show how the latter as well as apparent fat-tailed distributions can occur only as artifacts of the sliding interval analysis.

fat tails | Fokker–Planck equation | Langevin simulations

Arguably the most important problem in quantitative finance is to understand the nature of stochastic processes that underlie market dynamics. One aspect of the solution to this problem involves determining characteristics of the distribution of fluctuations in returns. Empirical studies conducted over the last decade have reported that they are non-Gaussian, scale in time, and have power law (or fat) tails (1–6). However, by combining increments at multiple times in their statistical analyses (sliding interval techniques), these studies implicitly assume that the stochastic process has stationary increments. For financial markets, it is not clear whether this assumption is valid. For example, it is possible that trading activity at the beginning of a trading day may differ from that at the end of the day. How is it possible to test whether intraday fluctuations are time-independent? If they are time-dependent, how can statistical analyses be conducted? Will results from previous studies be invalidated?

Our analysis is conducted on intraday Euro–Dollar exchange rates (traded 24 h per day) during 1999–2004 recorded in 1-min intervals. It is based on the assumption that intraday variations in the market follow the same underlying stochastic process every day. Then, a statistical analysis for fluctuations at a given time of the day can be conducted by using data from multiple trading days within the sample.

We find from this analysis the following. (i) The stochastic process is time-dependent and there are several intervals during the day where the standard deviation of increments exhibits power law behavior. Stochastic dynamics during these intervals is given by variable diffusion processes (2). (ii) Dynamical scaling indices and empirical scaling functions within these scaling intervals are different from previously reported results. We show how the latter can result from the application of sliding interval methods to a time-dependent stochastic process. (iii) Autocorrelation functions for variable diffusion processes exhibit anomalous characteristics similar to those reported in financial markets.

Results and Discussion

As in most studies of dynamics of financial markets, we assume that the return $\log P(t)$, where $P(t)$ represents the Euro–Dollar

rate at time t , executes stochastic dynamics. Since there are autocorrelations in $P(t)$ for time differences smaller than 10 min, we analyze the underlying stochastic process using increments $\varepsilon'(t) \equiv [\log P(t + 10) - \log P(t)]$ of the return. The analysis presented below is predicated on the assumption, for which we provide evidence, that the stochastic dynamics of $\varepsilon'(t)$ is the same between trading days. Then, we find that the average movement $\langle \varepsilon'(t) \rangle$ taken over the $\approx 1,500$ trading days during 1999–2004 nearly vanishes for each value of t . For the rest of our analysis, we remove this mean and study $\varepsilon(t) = \varepsilon'(t) - \langle \varepsilon'(t) \rangle$.

We first show that increments in return are time-dependent. Fig. 1*a* shows the behavior of the standard deviation $\sigma(t) \equiv \sqrt{\langle \varepsilon(t)^2 \rangle}$ of the Euro–Dollar rate as a function of the time of day. If the stochastic process were time-independent, the curve would be flat. Instead, $\sigma(t)$ exhibits complicated behavior and changes by more than a factor of 3 during the day. Hence, increments in the Euro–Dollar rate during a day are nonstationary. It has been proposed that this time dependence can be partially removed by using “tick-time” instead of clock-time (7).

To validate the assumption of daily repetition of the stochastic process, we implement a corresponding analysis of fluctuations throughout a trading week (8). Fig. 1*b* shows the standard deviation of returns averaged over the 300 weeks studied. The approximate daily periodicity of $\sigma(t)$ is evident, thereby justifying our approach. Similar observations were made on price increments for Euro–Dollar rate in ref. 8.

$\sigma(t)$ scales as power laws in time during several intervals within the day. Power-law fits to the data in some of these intervals are shown by colored lines in Fig. 1*a*. We focus our analysis on the time interval **I**, which begins at 9:00 a.m. New York time and lasts ≈ 3 h. The data shown in red in Fig. 2*a* show that within this interval, $\sigma(t)$ scales like $t^{-\eta}$, where t is measured from the beginning of **I** and the index $\eta = 0.13 \pm 0.04$. This scaling extends for more than 1.5 decades in time. Note that the value of η is different for the other scaling intervals. Similar variation in scaling exponents during the day has been reported previously (9).

The scaling index within **I** does not change significantly during the 6 years studied. This is demonstrated by independently analyzing three periods 1999–2000, 2001–2002, and 2003–2004. Fig. 2*b* shows that the scaling index remains nearly unchanged between these 2-year periods.

We have also analyzed the behavior of other moments $\langle |\varepsilon(t)|^\beta \rangle^{1/\beta}$ of fluctuations of returns. Fig. 2*a* shows that each of the moments $\beta = 0.5, 1.0, 2.0,$ and 3.0 also scales as a power law in time, and furthermore that the scaling index for each of them is consistent with the value of $\eta = 0.15$. This nearly uniform scaling of moments suggests that the return distribution itself scales in time. To provide a scaling ansatz, consider the stochastic variable

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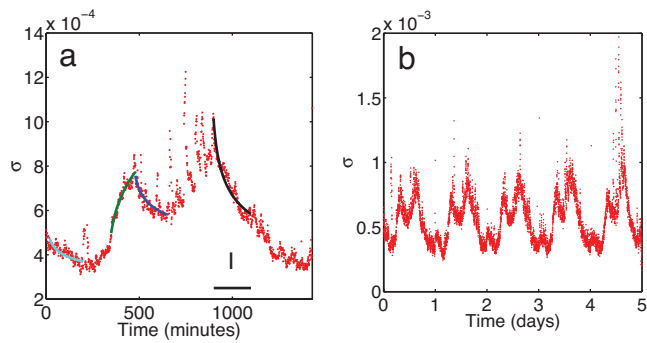


Fig. 1. Intraday increments in the Euro–Dollar exchange are nonstationary. (a) Standard deviation $\sigma(t) \equiv \sqrt{\langle \varepsilon(t)^2 \rangle}$ of the increments of the daily Euro–Dollar exchange as a function of the time of day (in Greenwich Mean Time). Our statistical analysis assumes that $\varepsilon(t)$ follows the same stochastic process each trading day. The average indicated by the brackets $\langle \cdot \rangle$ is taken over the $\approx 1,500$ trading days between 1999 and 2004, and the standard error at each point is typically 3%. Note that, if the stochastic dynamics had stationary increments, $\sigma(t)$ would be constant. Instead, it varies by more than a factor of 3 during the day, thus showing explicitly that the exchange rate has nonstationary increments. Notice also that $\sigma(t)$ scales in time during several intervals, four of which are highlighted by colored lines that are power-law fits. Our analysis focuses on the interval I shown by the horizontal solid line. (b) Weekly behavior of $\sigma(t)$ for the same data. Observe that it exhibits an approximate daily periodicity, thereby justifying our assumption of the daily repeatability of the stochastic process underlying the Euro–Dollar exchange rate.

$$x(\tau; t) = \sum_{k=0}^{N-1} \varepsilon(t + 10k),$$

which is the total increment of the return in a time interval $\tau = 10N$ starting from t . Denote its distribution by $W(x, \tau; t)$, where the final argument reiterates that the distribution can depend on the starting time of the interval. Our scaling ansatz is

$$W(x, \tau; 0) = \frac{1}{\tau^H} \mathcal{F}(u), \quad [1]$$

where H is the scaling index, $u = x/\tau^H$ is the scaling variable, and \mathcal{F} is the scaling function. Note that it is for a time interval starting from the beginning of I.

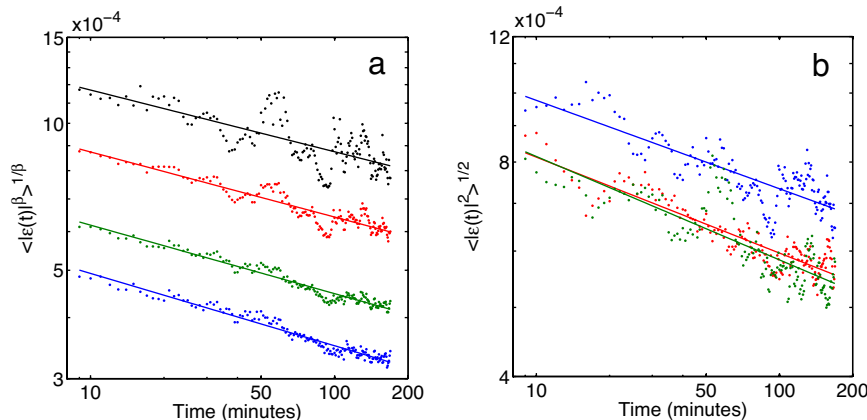


Fig. 2. Moments of increments scale in time. (a) log–log plot of $\langle |\varepsilon(t)|^\beta \rangle^{1/\beta}$ for $\beta = 0.5, 1.0, 2.0,$ and 3.0 , demonstrating power-law decay $t^{-\eta}$ for each index. Here, t is measured in local New York time stating at 9:00 a.m. The data for $\beta = 0.5, 1.0, 2.0,$ and 3.0 , shown in blue, green, red, and black, respectively, have scaling indices (given by the slopes of the solid lines) $\eta = 0.15 \pm 0.02, 0.14 \pm 0.02, 0.13 \pm 0.04,$ and 0.13 ± 0.08 . All of these values are consistent with $\eta \approx 0.15$, and hence a dynamical scaling index of $H = \frac{1}{2} - \eta \approx 0.35$. The error estimates on the exponents are the standard errors from the nonlinear fit including the standard deviations for each time point but neglecting any correlations between them. (b) Behavior of the standard deviation $\sigma(t)$ in the interval I during each of the periods 1999–2000 (blue), 2001–2002 (red), and 2003–2004 (green). The scaling index from nonlinear fits for the three data sets are $0.13 \pm 0.06, 0.14 \pm 0.04,$ and 0.14 ± 0.07 . The near equality of these indices shows that the scaling index is nearly invariant over time.

In addition to scaling, stochastic dynamics of the Euro–Dollar rate appears to have no memory. This can be demonstrated by evaluating the autocorrelation function

$$A(t_1, t_2) = \frac{\langle \varepsilon(t_1)\varepsilon(t_2) \rangle}{\sigma(t_1)\sigma(t_2)}.$$

We find that $A(t_1, t_2) = 1$ if $t_1 = t_2$, and of the order of 10^{-3} when $|t_1 - t_2| \geq 10$. This observation eliminates fractional Brownian motion (10) as a description for the underlying stochastic dynamics and strongly indicates that $\varepsilon(t)$ is a Markov process. Consequently, $x(\tau; 0)$ follows a Markov process in τ and $\partial W(x, \tau; 0)/\partial \tau$ depends only on $x(\tau; 0)$ and τ . If, in addition, $W(x, \tau; 0)$ has finite variance (see Fig. 4), it has been analytically established that the evolution of $W(x, \tau; 0)$ is given by a diffusion equation (11, 12)

$$\frac{\partial}{\partial \tau} W(x, \tau; 0) = \frac{1}{2} \frac{\partial^2}{\partial x^2} [D(x, \tau)W(x, \tau; 0)], \quad [2]$$

where $D(x, \tau)$ is the diffusion coefficient. There is no drift term in Eq. 2 because $x(\tau; 0)$ has zero mean. Note that the stochastic dynamics is completely determined by the diffusion coefficient, which, as shown below, depends on H . Hence, H can be considered to be the dynamical scaling index.

Because we have found scaling, consider solutions to Eq. 2 of the form given by Eq. 1. When $H = \frac{1}{2}$, the diffusion coefficient has been shown to be a function of u ; i.e., $D(x, \tau) = \mathcal{D}(u)$ (12). If, in addition, $\mathcal{D}(u)$ is symmetric in u , it is related to the scaling function by $\mathcal{F}(u) = D(u)^{-1} \exp(-\int^u dy y/D(y))$ (12, 13). When $H \neq \frac{1}{2}$, we can “rescale” time intervals by $\tilde{\tau} = \tau^{2H}$ (8, 14). In $\tilde{\tau}$, the stochastic process has a scaling index $\frac{1}{2}$ and a diffusion coefficient of the form $\mathcal{D}(x/\sqrt{\tilde{\tau}})$. Converting back to τ , $D(x, \tau) = 2H\tau^{2H-1} \mathcal{D}(u)$ (14).

Statistical analyses of financial markets have often been conducted by using sliding interval methods (2–6, 8, 15, 16), which implicitly assume that the underlying stochastic process has stationary increments. For example, they compute the distribution $W_S(x, \tau) = \langle W(x, \tau; t) \rangle_t$, where $\langle \cdot \rangle_t$ indicates an average over t . Many of these studies have reported that $W_S(x, \tau)$ scales as

$$W_S(x, \tau) = \frac{1}{\tau^{H_S}} \mathcal{F}_S(v), \quad [3]$$

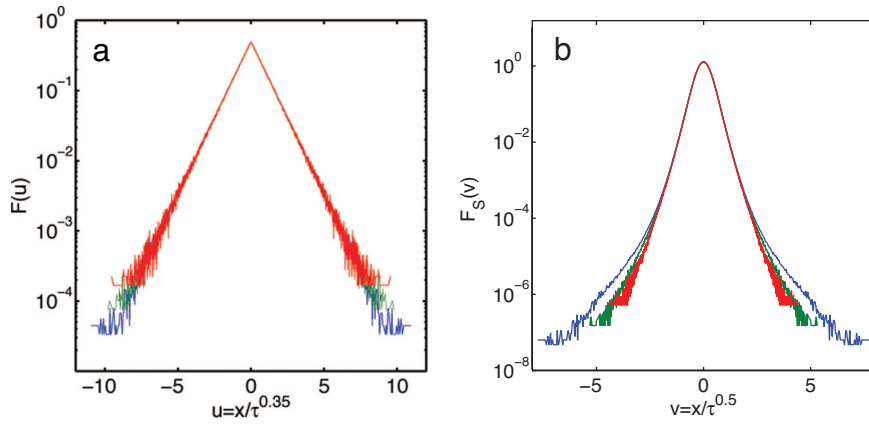


Fig. 3. Model data analysis demonstrating the appearance of spurious fat tails by means of a sliding interval technique. (a) Scaling function of the return distribution \mathcal{F} calculated from a collapse of data for $\tau = 10$ (blue), 100 (green), and 1,000 (red) units. The results are from a set of 5,000,000 independent stochastic processes with variable diffusion. The scaling index used was $H = 0.35$, and the diffusion coefficient was $2Ht^{2H-1}(1 + |u|)$. Note that \mathcal{F} is biexponential, as discussed in the text. (b) Sliding interval scaling function \mathcal{F}_S calculated from the same runs. Shown are results for sliding intervals with $\tau = 10$ (blue), 100 (green), and 1,000 (red) units from runs of length 10,000 units. Unlike \mathcal{F} , it appears to have fat tails. The scaling index used here for which the scaling collapse is achieved is $H_S = 1/2$ even though the dynamical scaling index is $H = 0.35$. Note, however, that although the central part of the distribution scales well, the tails do not.

where $v = x/\tau^{H_S}$ and $H_S \approx 1/2$. It has also been reported that the scaling function \mathcal{F}_S has power-law (or fat) tails (5, 6). However, it is important to understand that $W_S(x, \tau)$ is a solution of Eq. 2 only when the stochastic process is not time-dependent, in which case $H = H_S = 1/2$. In general, H_S and $W_S(x, \tau)$ are different from H and $W(x, \tau; 0)$. Next, we give an explicit example where this is the case, and, in addition, $W_S(x, \tau)$ appears to have fat tails even though $W(x, \tau; 0)$ does not.

Consider a diffusive process initiated at $x = 0$ that has a variable diffusion coefficient $2Ht^{2H-1}(1 + |u|)$. Its distribution has a scaling index H and a scaling function $\mathcal{F}(u) = 1/2 \exp(-|u|)$ (12, 13). (See the discussion after Eq. 2.) Numerical integration of the stochastic process for $H = 0.35$ confirms this claim (see Fig. 3a). In contrast, $W_S(x, \tau)$ calculated from the same data appears to scale with an index $H_S = 1/2$. Unlike \mathcal{F} , which is biexponential, the apparent scaling function \mathcal{F}_S (shown in Fig. 3b) has broad tails. However, a careful analysis reveals that distributions $W_S(x, \tau)$ do not scale in the tail region, and hence that \mathcal{F}_S is not well defined. Differences analogous to those between H and H_S have been noted for Lévy processes (17) and for the R/S analysis of Tsallis distributions (15).

The behavior of $\sigma(t)$ (Fig. 2a) can be calculated for variable diffusion processes. Assuming that τ is small, Ito calculus gives $\delta x^2 \equiv x(\tau; t)^2 = D(x, t)\tau + O(\tau^2)$. Averaging over returns at t gives

$$\langle \delta x^2 \rangle \approx \left[\int dx W(x, t; 0) D(x, t) \right] \tau. \quad [4]$$

In a variable diffusion process, $W(x, t; 0) = t^{-H} \mathcal{F}(u)$ and $D(x; t) = 2Ht^{2H-1} \mathcal{D}(u)$; consequently

$$\sqrt{\langle \delta x^2 \rangle} \sim t^{H-1/2}, \quad [5]$$

independent of the exact form of $\mathcal{D}(u)$. Results for the Euro-Dollar rate within the interval \mathbf{I} (Fig. 2a) that showed that $\eta \approx 0.15$ are therefore consistent with a scaling index $H = 1/2 - \eta \approx 0.35$. Note that, unlike for Lévy processes and fractional Brownian motion, $H < 1/2$ and is significantly lower than H_S reported in previous analyses of the Euro-Dollar exchange rate (between 0.5 and 0.6) (8, 16, 18). A general calculation for the moments of a variable diffusion process gives

$$\langle |\delta x|^\beta \rangle^{1/\beta} \sim t^{H-1/2}, \quad [6]$$

for all β , consistent with results shown in Fig. 2a.

To estimate H_S for an arbitrary variable diffusion process, we take the time average of the ensemble average of Eq. 4, giving

$$\langle \langle x(\tau; t)^2 \rangle \rangle_t \approx \left\langle \int dx W(x, t; 0) D(x, t) \right\rangle_t \tau. \quad [7]$$

Higher-order corrections to this approximation are small when $\tau \ll t$, a condition that is true for most intervals of length τ in a sliding interval calculation. Hence, $\langle \langle x(\tau; t)^2 \rangle \rangle_t \sim \tau$. Consequently, $H_S = 1/2$ regardless of the value of H .

Finally, we introduce a method to extract the empirical scaling function \mathcal{F} from the Euro-Dollar time series. Unfortunately, the available data are insufficient to determine $\mathcal{F}(u)$ accurately using the usual method of collapsing $W(x, \tau; 0)$ for each value of τ . However, since we have determined H (≈ 0.35) independently, we can use Eq. 1 for multiple values of τ in the interval \mathbf{I} (i.e., τ between ≈ 10 and 160 min) to determine \mathcal{F} . The result is shown in Fig. 4a. Note that the distribution has an approximate biexponential form. Since exponential distributions have finite variance, all assumptions needed for the derivation of Eq. 2 are justified. However, it is asymmetric and decays more slowly on the negative side. By contrast, the empirical sliding interval scaling function $\mathcal{F}_S(v)$ for the same time interval is shown in Fig. 4b. For this case, the scaling collapse is achieved for $H_S = 1/2$. $\mathcal{F}_S(v)$ has broader tails, consistent with previous reports (6, 18). However, in light of the example discussed earlier and the fact that $H \neq 1/2$, it is unlikely that \mathcal{F}_S is well defined for this financial market data within the interval \mathbf{I} .

Variable diffusion processes exhibit another signature (stylized fact) of market fluctuations. Since they have no drift, the autocorrelation functions of these increments vanish. However, a small fraction of these random walks reach anomalously high values of $|x|$ and hence experience large diffusion rates. Consequently, they execute large movements (whose directions are uncorrelated) repeatedly. As a result, the autocorrelation function for the signal $|x(t)|$ [or for the signal $x(t)^2$] will decay slowly in t . Such behavior, referred to as “clustering of volatility” is seen in the Euro-Dollar exchange rate and has been reported in empirical studies of other financial markets (19–21).

Conclusions

We have shown that the stochastic process underlying intraday fluctuations in the Euro-Dollar exchange rate is time-dependent

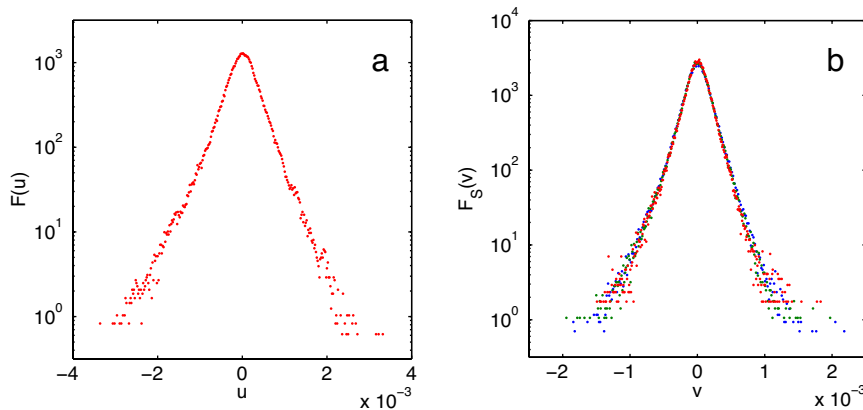


Fig. 4. Foreign exchange data analysis demonstrating the appearance of spurious fat tails by means of a sliding interval technique. (a) Empirical scaling function \mathcal{F} for interval I calculated assuming the scaling ansatz Eq. 1 with $H = 0.35$ and values of τ between 10 and 160 min. Note that \mathcal{F} is slightly asymmetric and approximately biexponential. Since exponential distributions have finite variance, all assumptions needed for the derivation of Eq. 2 are justified. (b) Empirical sliding interval scaling function \mathcal{F}_S for interval I calculated by scaling collapse of data using the ansatz Eq. 3 for τ of 10 (blue), 20 (green), and 40 (red) min. Note that the tails of \mathcal{F}_S are broader.

and that there are several intervals during which the standard deviation of increments exhibits scaling. The stochastic dynamics within these scaling intervals was shown to be diffusive with a diffusion coefficient that depends on both time and the exchange rate (2, 12). We presented a detailed analysis of one of the scaling regions that begins at 9:00 a.m. New York time and last for ≈ 3 h. The dynamical scaling index for the variable diffusion process here was shown to be ≈ 0.35 , significantly lower than the sliding interval value $H_S \approx 0.5$ reported in previous analyses of financial markets. In addition, unlike previous reports of fat-tailed distributions, the empirical scaling function within the interval has exponential tails. We showed that these discrepancies can result from the inappropriate use of sliding interval techniques to study stochastic processes with nonstationary increments.

The analysis given here applies to stochastic dynamics of a single scaling interval. However, daily fluctuations in the Euro–Dollar rate are a combination of scaling intervals with distinct indices, and possibly regions with no scaling. We have not yet determined how to extend our analysis beyond a single scaling region. Because of this, it is not clear how to interpret the distributions over intervals longer than a scaling region, including inter-day data.

Materials and Methods

We analyzed 1-min-interval tick data of the Euro–Dollar exchange rate in the 6-year period 1999–2004. The data were obtained from Olsen Financial Technologies, Zürich, Switzerland, and consisted of the closing bid and ask values for each minute interval, 24 h per day. The price used in our computations was the mean of the bid and ask values; instances where one

or both of them were not available were discarded from consideration.

Our analysis was limited to days in the 6-year period that the New York markets were open. Specifically, Euro–Dollar rates for national holidays in the United States; September 11–14, 2001, after attacks on the World Trade Center; and the day of observance of President Reagan’s funeral, June 11, 2004, were not used in the computations.

The 24-h results of Fig. 1 are plotted as a function of Greenwich Mean Time (GMT). In conducting the analysis for the interval I, we note that it lies within the period when banks and the stock exchange in New York are open. Although we do not have the data on trading volume to corroborate it, we assume that currency trading during this period is dominated by trading activity in New York. Consequently, in generating Fig. 2, we shifted time to account for the conversions between Eastern Standard Time (EST) and Eastern Daylight Time (EDT). The origin of time in Fig. 2 is set to 9:00 a.m. New York time. The data used in Fig. 4 are the same as those used in Fig. 2. The simulated data shown in Fig. 3 was generated by Langevin integration assuming Ito stochastic noise. That is, $dx = \sqrt{D(x, t)}dB$, where dB is a normally distributed random variable over the time interval dt . The value of the diffusion coefficient $D(x, t)$ at the beginning of the time interval was used during the entire time interval (Ito calculus). For the results shown in the figure, $dt = 10^{-4}$.

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